Large-θ_{13} Perturbation Theory of Neutrino Oscillation for Long-Baseline Experiments

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Abstract

The Cervera *et al.* formula, the best known approximate formula of neutrino oscillation probability for long-baseline experiments, can be regarded as a second-order perturbative formula with small expansion parameter $\epsilon \equiv \Delta m^2_{21}/\Delta m^2_{31} \simeq 0.03$ under the assumption $s_{13} \simeq \epsilon$. If θ_{13} is large, as suggested by a candidate ν_e event at T2K as well as the recent global analyses, higher order corrections of s_{13} to the formula would be needed for better accuracy. We compute the corrections systematically by formulating a perturbative framework by taking θ_{13} as $s_{13} \sim \sqrt{\epsilon} \simeq 0.18$, which guarantees its validity in a wide range of θ_{13} below the Chooz limit. We show on general ground that the correction terms must be of order ϵ^2. Yet, they nicely fill the mismatch between the approximate and the exact formulas at low energies and relatively long baselines. General theorems are derived which serve for better understanding of δ-dependence of the oscillation probability. Some interesting implications of the large θ_{13} hypothesis are discussed.

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I. INTRODUCTION

One of the most important progresses in particle physics in the last decades is the discovery of neutrino masses [1] and the lepton flavor mixing [2]. It was done through observing neutrino oscillation phenomena and it constitutes, up until this moment, only available experimental method for measuring lepton mixing parameters. Δm^2_{32} and θ_{23} are determined by atmospheric neutrino observation by Super-Kamiokande [3–5], and then by accelerator neutrino experiments [6, 7]. Δm^2_{21} and θ_{12} are measured independently by two types of experiments, the KamLAND reactor experiment [8–10] and the solar neutrino observation using various experimental techniques. For the latest results and for a review of the solar neutrino experiments see e.g., [11, 12] and [13], respectively. The remaining mixing angle θ_{13} is being explored by the ongoing and the upcoming accelerator [14, 15] and reactor neutrino experiments [16–18]. If it turned out that θ_{13} is not too small, we may proceed to measure CP violation by the lepton Kobayashi-Maskawa (KM) [19] phase δ_ℓ, to which we refer just δ in this paper.

It is expected that precision measurement is required to determine δ because CP violation effect is tiny due to suppression by the two small factors, $\Delta m^2_{21}/\Delta m^2_{31}$ and the Jarlskog coefficient $J \equiv c_{12}s_{12}c_{23}s_{23}c^2_{13}s_{13}$ [20]. Therefore, understanding of full complexity of neutrino oscillation phenomena would be of some help e.g., to design future experiments. An example of such is the parameter degeneracy [21–23], the problem of multiple copy of the solutions of mixing parameters allowed by given sufficient but limited numbers of experimental data. See [24] for a comprehensive overview of this phenomenon. To facilitate understanding of qualitative features of the neutrino oscillation, it is crucially important to have analytic formula, albeit approximate, for the oscillation probability. For relatively short baseline experiments, such as low-energy superbeam [25–27], the matter perturbation theory works [28, 29]. So far, most of the analyses for long baseline of $L \gtrsim 1000$ km were done by using the well known Cervera *et al.* formula [30].[1]

A simple way of deriving the Cervera *et al.* formula is to expand the exact oscillation probability by small expansion parameters, $\epsilon \equiv \Delta m^2_{21}/\Delta m^2_{31}$ and $s_{13} \equiv \sin\theta_{13}$, both to second order. While the former is known to be small $\epsilon \simeq 0.03$, the latter can be larger by almost an order of magnitude; Currently, it is only bounded from above by the Chooz limit, $s_{13} \lesssim 0.18$ [34–37]. We note that there is an indication for nonzero θ_{13} comparable to the Chooz limit from global fit of the solar, reactor, atmospheric, and the accelerator experiments [10, 12, 38–42]. Though the statistical significance of the indication is not high enough to be compelling, it certainly gives a good motivation for examining effects of such large values of θ_{13}. Recently, a candidate event for ν_e appearance has been seen in the T2K experiment [43] which further strengthens the motivation for taking the large-θ_{13} hypothesis seriously.

If θ_{13} is large, higher order terms of s_{13} would be needed to achieve better agreement with the exact oscillation probability. In this paper, we compute such higher order corrections of s_{13}. To facilitate systematic computation we formulate a perturbative framework by assuming s_{13} as large as $\simeq \sqrt{\epsilon}$, which is comparable to the Chooz limit, and by taking the case of long enough baseline where the matter effect is comparable to the vacuum one. We

[1] However, it was shown that analysis of the parameter degeneracy with the matter-perturbative formula has been proved to give a transparent view of the phenomenon [31–33], such as the decoupling between the degeneracies.

call this framework as the "$\sqrt{\epsilon}$ perturbation theory" as opposed to the ϵ perturbation theory for the Cervera *et al.* formula as named in [33]. We derive the second order formulas for the oscillation probabilities in all channels. By taking the ansatz, $s_{13} \simeq \sqrt{\epsilon}$, our formulas will be valid for a wide range of θ_{13}, $\epsilon \lesssim s_{13} \lesssim \sqrt{\epsilon}$. The $\sqrt{\epsilon}$ perturbation theory has been formulated earlier for relatively short baseline setting [44].

Some characteristic features of the computed formulas prompted us to think about general property of the oscillation probability, which resulted in the two general theorems described in Sec. II. For example, we show that the δ dependent terms in the ν_e-related oscillation probabilities exist only in odd terms of s_{13}, and conversely, all the terms odd in s_{13} have δ dependence. Thanks to the theorems, quite (un)interestingly, we can prove on general ground that the correction terms to the Cervera *et al.* formula have to be second order in ϵ, eliminating the possibility of having large corrections. Yet, we will observe that at super long baseline, $L = 4000$ km, our correction terms nicely fill a sizable gap between the exact oscillation probability and the Cervera *et al.* formula, which exists in a limited range of energy for θ_{13} comparable to the Chooz limit.

The possible large value of θ_{13} could generate some interesting effects. For example, there arise terms of order $\epsilon^{5/2}$ consisting solely of δ-dependent terms in the ν_e appearance oscillation probability, which is smaller only by a factor of $\simeq 5$ compared to the existing terms in the Cervera *et al.* formula. Another intriguing feature arises when the non-standard interactions (NSI) of neutrinos are included into the system. For an extensive list of the references for NSI including the original ones see, e.g., [45]. Some of the NSI dependent terms get enhanced by large θ_{13}, and decoupling of some NSI elements from the ν_e-related appearance probabilities no more hold. On the other hand, the smallness of the correction terms to the Cervera *et al.* formula has a consequence that, quite naturally, their influence to parameter determination including the issue of parameter degeneracy is quite limited.

Following the discussion of general property of oscillation probability in Sec. II, we formulate our $\sqrt{\epsilon}$ perturbation theory (Sec. III), and derive the expressions of the oscillation probabilities valid to order ϵ^2 in $\nu_e \to \nu_\mu$ and $\nu_\mu \to \nu_\mu$ channels (Sec. IV). The characteristic features of the formulas are shown to be understood thanks to the theorems given in Sec. II. The accuracy of our formula is checked against the exact results for a large value of θ_{13} (Sec. V). Then, we calculate order $\epsilon^{5/2}$ terms of the oscillation probability in the $\nu_e \to \nu_\mu$ channel which could be relevant for measurement of δ in its extreme precision (Sec. VI). Implications of smallness of the large θ_{13} corrections are explored by treating the parameter degeneracy (Sec. VII). The second-order formula of the oscillation probability in the ν_e-related appearance channels is derived for systems with NSI (Sec. VIII). Concluding remarks are given in Sec. IX. Appendices are devoted for a proof of a general theorem for suppression of CP phase effect (Appendix A), the explicit expressions of the S matrix elements in the θ_{23}-rotated basis (Appendix B), and explicit forms of the oscillation probabilities in the remaining channels (Appendix C).

II. GENERAL FEATURES OF NEUTRINO OSCILLATIONS IN MATTER

Before constructing perturbative framework of neutrino oscillation, let us discuss some generic features of the oscillation probability. They are interesting by themselves, and statements about $\cos\delta$ dependence do not appear to be explicitly spelled out in the literature to our knowledge. It will also help us to understand characteristic features of the perturbation theory of neutrino oscillation to be discussed in the rest of this paper. An example of this

is that CP phase effect appear only in terms of half-integral order of the small expansion parameter ϵ in our $\sqrt{\epsilon}$ perturbation theory.

We note that the general features of neutrino oscillation probability which exactly hold with [46, 47] or without [48] constant matter density approximation can be used to prove some useful properties of the oscillation probabilities. That is, we call the readers' attention to the following two theorems:

- Theorem A: In ν_e-related oscillation probabilities the δ-dependence exists only in odd terms in s_{13}. Conversely, all the odd terms in s_{13} are accompanied with either $\cos\delta$ or $\sin\delta$.

- Theorem B: δ-dependent terms in the oscillation probability in matter, not only $\sin\delta$ but also $\cos\delta$ terms, must come with the two suppression factors, $\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ and the reduced (or "not quite") Jarlskog coefficient $J_r \equiv c_{12}s_{12}c_{23}s_{23}s_{13}$.

In talking about s_{13}- and δ-dependences we assume the standard form of the lepton flavor mixing matrix, the MNS matrix [2] given in (7), the convention which will be used throughout this paper.

To prove the theorem A we note that s_{13} and δ enter into the Hamiltonian through the single variable $z \equiv s_{13}e^{i\delta}$. Therefore, the oscillation probability P can be written as a power series expansion as $P = \sum_{n,m}^{\infty} f_{nm} z^n (z^*)^m$, where $f_{nm} = f^*_{mn}$ for reality of P. On the other hand, the result obtained in [48] by extending discussions in [49] says that there are only $\cos\delta$ and $\sin\delta$ terms in ν_e related oscillation probabilities, and no higher harmonics of δ. It means that only the terms that satisfy $m = n \pm 1$ survive. It leaves the unique form of the oscillation probability $P = K(s^2_{13})s_{13}\cos\delta + M(s^2_{13})s_{13}\sin\delta$, where K and M are some functions. At the same time, this construction guarantees that all the odd terms in s_{13} are accompanied by δ. This is nothing but the theorem A.

Now, let us discuss the theorem B. If the adiabaticity holds existence of the suppression factors by the Δm^2 ratio and the angle factors for $\sin\delta$ terms is well known. In fact, one can show that the latter is precisely the Jarlskog coefficient $J \equiv c_{12}s_{12}c_{23}s_{23}c^2_{13}s_{13}$ thanks to the Naumov identity [50, 51]

$$\Delta \bar{m}^2_{12} \Delta \bar{m}^2_{23} \Delta \bar{m}^2_{31} \bar{J} \sin\bar{\delta} \;=\; \Delta m^2_{12} \Delta m^2_{23} \Delta m^2_{31} J \sin\delta, \tag{1}$$

where the quantities with over-bar in the left-hand side denote the ones in matter which correspond to the one in vacuum in the right-hand side of (1). Notice that $\cos\bar{\delta}$ does not contain $\sin\delta$ [47], and the proportionality between $\sin\bar{\delta}$ and $\sin\delta$ is also guaranteed by the Toshev identity $\bar{c}_{23}\bar{s}_{23}\sin\bar{\delta} = c_{23}s_{23}\sin\delta$ [52]. The equation (1) indicates that the $\sin\delta$ term vanishes in the limit of one of $\Delta m^2_{ij} \to 0$, or vanishing limit of one of the mixing angles.

Whether the same statement apply to $\cos\delta$ terms or not, or if the theorem B is valid at all for cases with generic matter density profiles, is not obvious. Apparently no general statement has been made in the literature. Vanishing of any δ dependence (including cosine) in the absence of Δm^2_{21} can be proved similarly as the phase reduction theorem given in [33] as a special case of turning off the non-standard interactions. For constant matter density proportionality of $\cos\delta$ terms to the Jarlskog coefficient J (reduced coefficient J_r) in the ν_e-related channels (oscillation channels in the $\nu_\mu - \nu_\tau$ sector) is explicitly proved in [47] by

deriving the exact forms of the oscillation probabilities.[2] Therefore, what is left is to show that the last statement holds under arbitrary matter density profile without recourse to the assumption of adiabaticity. The proof for this general case is described in Appendix A.

III. FORMULATING LARGE-θ_{13} PERTURBATION THEORY

In this section, we formulate a large-θ_{13} perturbation theory of neutrino oscillation. We use an ansatz

$$s_{13} \simeq \sqrt{\epsilon}, \qquad \epsilon \equiv \frac{\Delta m^2_{21}}{\Delta m^2_{31}} \simeq 0.03 \tag{2}$$

to formulate our perturbative framework, hence the name "$\sqrt{\epsilon}$ perturbation theory". It implies $\sin^2 2\theta_{13} \simeq 0.12$, the value comparable with the Chooz limit. We work in anticipation of long baselines of several thousand kilometers, so that $r_A \equiv a/\Delta m^2_{31} \sim \mathcal{O}(1)$, and $\Delta m^2_{31} L/E \sim \mathcal{O}(1)$ to be not far from the first oscillation maximum. Hereafter, $a \equiv 2\sqrt{2} G_F N_e(x) E$ is a coefficient for measuring the matter effect on neutrinos propagating in medium of electron number density $N_e(x)$ [53], where G_F is the Fermi constant and E is the neutrino energy. In perturbative calculation of the oscillation probability we take constant electron number density approximation.

If we take the ansatz $s_{13} \simeq \epsilon$ (which corresponds to $\sin^2 2\theta_{13} \simeq 4\epsilon^2 \simeq 4 \times 10^{-3}$) instead of (2), we obtain the widely used Cervera *et al.* formula [30] by keeping terms to order ϵ^2. Of course, there may exist the other small (or large) parameters, such as $\Delta m^2_{31} L/E$ and $a/\Delta m^2_{31}$ at short baselines, depending upon the experimental settings. The similar $\sqrt{\epsilon}$ perturbation theory for shorter baselines is discussed by assuming $r_A \sim \sqrt{\epsilon}$ [44].

The remaining potentially small parameters would be $\pi/4 - \theta_{23}$, but we do not take it as an expansion parameter for two reasons:[3] (1) A rather large range is currently allowed for θ_{23}, and moreover the situation will not be changed even with the next generation experiments [57]. (2) As will become evident in formulating our perturbative framework θ_{23} is an "external parameter" which is irrelevant in doing perturbative computation.

We follow [33] to formulate the perturbative treatment of neutrino oscillation. The S matrix describes possible flavor changes after traversing a distance L,

$$\nu_\alpha(L) = S_{\alpha\beta} \nu_\beta(0), \tag{3}$$

and the oscillation probability is given by

$$P(\nu_\beta \to \nu_\alpha; L) = |S_{\alpha\beta}|^2. \tag{4}$$

When the neutrino evolution is governed by the Schrödinger equation, $i\frac{d}{dx}\nu = H\nu$, S matrix is given as

$$S = T\text{exp}\left[-i \int_0^L dx H(x)\right] \tag{5}$$

[2] The difference of whether the suppression factor is provided by J or J_r does not make difference in our following discussions. Hence, we do not discuss this point further apart from making a brief comment in Appendix A.

[3] Nonetheless, one can of course further expand the oscillation probability in terms of $\pi/4 - \theta_{23}$ assuming it small, as is done e.g., in [54–56].

where T symbol indicates the "time ordering" (in fact "space ordering" here). The right-hand side of (5) may be written as e^{-iHL} for the case of constant matter density. In the standard three-flavor neutrinos, Hamiltonian is given (with $a = 2\sqrt{2}G_F N_e E$) by

$$H = \frac{1}{2E}\left\{U\begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta m^2_{21} & 0 \\ 0 & 0 & \Delta m^2_{31}\end{bmatrix}U^\dagger + \begin{bmatrix} a(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\right\}, \tag{6}$$

where $\Delta m^2_{ji} \equiv m^2_j - m^2_i$. In (6) U is the MNS matrix and can be written in the standard notation as

$$U = U_{23}U_{13}U_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23}\end{bmatrix}\begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13}\end{bmatrix}\begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1\end{bmatrix} \tag{7}$$

with the notation $s_{ij} \equiv \sin\theta_{ij}$ etc. and δ being the lepton KM phase.

To formulate perturbative treatment it is convenient to work with the tilde basis defined as $\tilde{\nu}_\alpha = (U^\dagger_{23})_{\alpha\beta}\nu_\beta$, in which the Hamiltonian is related to the flavor basis one as [58]

$$\tilde{H} = U^\dagger_{23} H U_{23} \tag{8}$$

where U_{23} is defined in (7). The S matrix in the flavor basis is related to the S matrix in the tilde basis as

$$S(L) = U_{23}\tilde{S}(L)U^\dagger_{23} \tag{9}$$

where $\tilde{S}(L) = T\exp\left[-i\int_0^L dx\tilde{H}(x)\right]$. The unperturbed part of the tilde-basis Hamiltonian is given by

$$\tilde{H}_0 \ = \ \Delta\begin{bmatrix} r_A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{bmatrix}, \tag{10}$$

where $\Delta \equiv \frac{\Delta m^2_{31}}{2E}$ and $r_A \equiv \frac{a}{\Delta m^2_{31}}$. While the perturbed part is written with another simplified notation $r_\Delta \equiv \frac{\Delta m^2_{21}}{\Delta m^2_{31}}$ as

$$\begin{aligned}
\tilde{H}_1 \ &= \ \Delta\begin{bmatrix} 0 & 0 & s_{13}e^{-i\delta} \\ 0 & 0 & 0 \\ s_{13}e^{i\delta} & 0 & 0\end{bmatrix} + \Delta\begin{bmatrix} r_\Delta s^2_{12} + s^2_{13} & r_\Delta c_{12}s_{12} & 0 \\ r_\Delta c_{12}s_{12} & r_\Delta c^2_{12} & 0 \\ 0 & 0 & -s^2_{13}\end{bmatrix} \\
&- \ \Delta\begin{bmatrix} 0 & 0 & \left(r_\Delta s^2_{12} + \frac{1}{2}s^2_{13}\right)s_{13}e^{-i\delta} \\ 0 & 0 & r_\Delta c_{12}s_{12}s_{13}e^{-i\delta} \\ \left(r_\Delta s^2_{12} + \frac{1}{2}s^2_{13}\right)s_{13}e^{i\delta} & r_\Delta c_{12}s_{12}s_{13}e^{i\delta} & 0\end{bmatrix} \\
&- \ \Delta r_\Delta\begin{bmatrix} s^2_{12}s^2_{13} & \frac{1}{2}c_{12}s_{12}s^2_{13} & 0 \\ \frac{1}{2}c_{12}s_{12}s^2_{13} & 0 & 0 \\ 0 & 0 & -s^2_{12}s^2_{13}\end{bmatrix}.
\end{aligned} \tag{11}$$

The first, second, third, and the fourth terms in (11) are of order $\epsilon^{\frac{1}{2}}$, ϵ^1, $\epsilon^{\frac{3}{2}}$, and ϵ^2, respectively.

To calculate $\tilde{S}(L)$ perturbatively we define $\Omega(x)$ as $\Omega(x) = e^{i\tilde{H}_0 x}\tilde{S}(x)$, which obeys the evolution equation

$$i\frac{d}{dx}\Omega(x) = H_1\Omega(x) \tag{12}$$

where

$$H_1 \equiv e^{i\tilde{H}_0 x}\tilde{H}_1 e^{-i\tilde{H}_0 x} \tag{13}$$

Then, $\Omega(x)$ can be computed perturbatively as

$$\begin{aligned}\Omega(x) &= 1 + (-i)\int_0^x dx' H_1(x') + (-i)^2 \int_0^x dx' H_1(x') \int_0^{x'} dx'' H_1(x'') \\ &+ (-i)^2 \int_0^x dx' H_1(x') \int_0^{x'} dx'' H_1(x'') \int_0^{x''} dx''' H_1(x''') + \mathcal{O}(\epsilon^4).\end{aligned} \tag{14}$$

where the "space-ordered" form in (14) is essential because of the non-commutativity between H_1 of different locations. Having obtained $\Omega(x)$ $\tilde{S}$ matrix can be written as

$$\tilde{S}(x) = e^{-i\tilde{H}_0 x}\Omega(x). \tag{15}$$

The results of $\tilde{S}(x)$ matrix elements to second-order in ϵ are given in Appendix B. Then, the S matrix can be computed by making a rotation in (2-3) space $S = U_{23}\tilde{S}U_{23}^\dagger$ as in (9). (See (B9).) Finally, the oscillation probability can readily be obtained by using (4). For example, the one in the $\nu_e \to \nu_\mu$ channel can be given by using the $\tilde{S}$ matrix elements as

$$\begin{aligned}P(\nu_e \to \nu_\mu) &\equiv |S_{\mu e}|^2 = s_{23}^2 |\tilde{S}_{e\tau}^{(1/2)}(-\delta)|^2 + 2c_{23}s_{23}\mathrm{Re}\left[\tilde{S}_{e\tau}^{(1/2)}(-\delta)^* \tilde{S}_{e\mu}^{(1)}(-\delta)\right] \\ &+ c_{23}^2 |\tilde{S}_{e\mu}^{(1)}(-\delta)|^2 + 2s_{23}^2 \mathrm{Re}\left[\tilde{S}_{e\tau}^{(1/2)}(-\delta)^* \tilde{S}_{e\tau}^{(3/2)}(-\delta)\right].\end{aligned} \tag{16}$$

IV. PERTURBATIVE EXPRESSION OF THE OSCILLATION PROBABILITY

In this section, we present perturbative expressions of the oscillation probabilities in the ν_e related and the $\nu_\mu - \nu_\tau$ sectors to order ϵ^2 in our $\sqrt{\epsilon}$ perturbation theory. All the formulas in this section are given in the form as

$$P(\nu_\alpha \to \nu_\beta) = P_{\alpha\beta}^{(0)} + P_{\alpha\beta}^{(1)} + P_{\alpha\beta}^{(3/2)} + P_{\alpha\beta}^{(2)} \tag{17}$$

in which we use L to denote the baseline distance. Some order $\epsilon^{5/2}$ terms will be discussed in Sec. VI. The antineutrino probability can be obtained from the neutrino probability by the replacement as $P(\bar{\nu}_\alpha \to \bar{\nu}_\beta; \delta, a) = P(\nu_\alpha \to \nu_\beta; -\delta, -a)$. Similarly, the T-conjugate one is given by $P(\nu_\beta \to \nu_\alpha; \delta, a) = P(\nu_\alpha \to \nu_\beta; -\delta, a)$. Remember that the following abbreviated notations are used: $\Delta \equiv \frac{\Delta m_{31}^2}{2E}$, $r_\Delta \equiv \frac{\Delta m_{21}^2}{\Delta m_{31}^2}$, and $r_A \equiv \frac{a}{\Delta m_{31}^2}$.

A. Oscillation Probabilities in ν_e related sector

There is no zeroth oder term in the appearance channels. With use of the reduced Jarlskog coefficient $J_r \equiv c_{12}s_{12}c_{23}s_{23}s_{13}$, the order ϵ^1, $\epsilon^{3/2}$, and ϵ^2 terms in the oscillation probability in the $\nu_e \to \nu_\mu$ channel are given by

$$P_{e\mu}^{(1)} = 4s_{23}^2 s_{13}^2 \frac{1}{(1-r_A)^2} \sin^2 \frac{(1-r_A)\Delta L}{2}, \tag{18}$$

$$P_{e\mu}^{(3/2)} = 8J_r \frac{r_\Delta}{r_A(1-r_A)} \cos\left(\delta - \frac{\Delta L}{2}\right) \sin \frac{r_A \Delta L}{2} \sin \frac{(1-r_A)\Delta L}{2}, \tag{19}$$

$$\begin{aligned} P_{e\mu}^{(2)} &= 4c_{23}^2 c_{12}^2 s_{12}^2 \left(\frac{r_\Delta}{r_A}\right)^2 \sin^2 \frac{r_A \Delta L}{2} \\ &- 4s_{23}^2 \left[s_{13}^4 \frac{(1+r_A)^2}{(1-r_A)^4} - 2s_{12}^2 s_{13}^2 \frac{r_\Delta r_A}{(1-r_A)^3} \right] \sin^2 \frac{(1-r_A)\Delta L}{2} \\ &+ 2s_{23}^2 \left[2s_{13}^4 \frac{r_A}{(1-r_A)^3} - s_{12}^2 s_{13}^2 \frac{r_\Delta}{(1-r_A)^2} \right] (\Delta L) \sin(1-r_A)\Delta L. \end{aligned} \tag{20}$$

As is obvious from (B9), the similar expressions $P_{e\tau}^{(i)}$ for the $\nu_e \to \nu_\tau$ channel can be obtained from $P_{e\mu}^{(i)}$ in (18)-(20) by the transformation $c_{23} \to -s_{23}$ and $s_{23} \to c_{23}$. The explicit expressions are given in Appendix C. Given $P_{e\mu}^{(i)}$ and $P_{e\tau}^{(i)}$, $P_{ee}^{(i)}$ can be obtained by using the perturbative unitarity relation[4]

$$P_{ee}^{(i)} = \delta_{i,0} - P_{e\mu}^{(i)} - P_{e\tau}^{(i)} \qquad (i = 0, 1, 3/2, 2). \tag{21}$$

Therefore, we do not present their explicit forms. Notice that $P_{e\mu}^{(3/2)} + P_{e\tau}^{(3/2)} = 0$ as it should because there must be no δ dependent terms in P_{ee} [49, 60].

B. Oscillation Probabilities in $\nu_\mu - \nu_\tau$ sector

The order-by-order perturbative formulas of the oscillation probabilities in the $\nu_\mu \to \nu_\mu$ channel to order ϵ^2 can be computed via a similar manner. The results are given by

$$P_{\mu\mu}^{(0)} = 1 - 4c_{23}^2 s_{23}^2 \sin^2\left(\frac{\Delta L}{2}\right), \tag{22}$$

$$\begin{aligned} P_{\mu\mu}^{(1)} &= -4s_{23}^4 s_{13}^2 \frac{1}{(1-r_A)^2} \sin^2 \frac{(1-r_A)\Delta L}{2} \\ &- 2c_{23}^2 s_{23}^2 \left[s_{13}^2 \frac{r_A}{1-r_A} - c_{12}^2 r_\Delta \right] (\Delta L) \sin \Delta L \\ &+ 4c_{23}^2 s_{23}^2 s_{13}^2 \frac{1}{(1-r_A)^2} \sin \frac{(1+r_A)\Delta L}{2} \sin \frac{(1-r_A)\Delta L}{2}, \end{aligned} \tag{23}$$

[4] In fact, we computed all the $P_{e\alpha}^{(i)}$ ($\alpha = e, \mu, \tau$) by the same procedure and explicitly verified the perturbative unitarity relation.

$$
\begin{aligned}
P_{\mu\mu}^{(3/2)} &= -8J_r \cos\delta \left(c_{23}^2 - s_{23}^2\right) \frac{r_\Delta r_A}{1-r_A} \sin^2\left(\frac{\Delta L}{2}\right) \\
&+ 8J_r \cos\delta \frac{r_\Delta}{r_A(1-r_A)} \sin^2\left(\frac{r_A \Delta L}{2}\right) \\
&- 16 J_r \cos\delta \; s_{23}^2 \frac{r_\Delta}{r_A(1-r_A)} \sin\frac{\Delta L}{2} \sin\frac{r_A \Delta L}{2} \cos\frac{(1-r_A)\Delta L}{2},
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
P_{\mu\mu}^{(2)} &= -c_{23}^2 s_{23}^2 \left(s_{13}^2 \frac{r_A}{1-r_A} - c_{12}^2 r_\Delta\right)^2 (\Delta L)^2 \cos\Delta L \\
&+ 2c_{23}^2 s_{23}^2 \left[s_{13}^4 \frac{r_A(1+r_A)}{(1-r_A)^3} - (c_{12}^2 + s_{12}^2 r_A^2) s_{13}^2 \frac{r_\Delta}{(1-r_A)^2} - c_{12}^2 s_{12}^2 \frac{r_\Delta^2}{r_A}\right] (\Delta L) \sin\Delta x \\
&+ 2c_{23}^2 s_{23}^2 \left[s_{13}^4 \frac{r_A}{(1-r_A)^3} + (c_{12}^2 - s_{12}^2) s_{13}^2 \frac{r_\Delta}{(1-r_A)^2}\right] (\Delta L) \sin r_A \Delta L \\
&- 2s_{23}^4 \left[2 s_{13}^4 \frac{r_A}{(1-r_A)^3} - s_{12}^2 s_{13}^2 \frac{r_\Delta}{(1-r_A)^2}\right] (\Delta L) \sin(1-r_A)\Delta L \\
&- 4c_{23}^2 s_{23}^2 \left[s_{13}^4 \frac{r_A(2+r_A)}{(1-r_A)^4} - 2 s_{12}^2 s_{13}^2 \frac{r_\Delta r_A}{(1-r_A)^3} - c_{12}^2 s_{12}^2 \left(\frac{r_\Delta}{r_A}\right)^2\right] \sin^2\frac{\Delta L}{2} \\
&+ 4\left[c_{23}^2 s_{23}^2 s_{13}^4 \frac{r_A(2+r_A)}{(1-r_A)^4} - 2c_{23}^2 s_{23}^2 s_{12}^2 s_{13}^2 \frac{r_\Delta r_A}{(1-r_A)^3} - c_{23}^4 c_{12}^2 s_{12}^2 \left(\frac{r_\Delta}{r_A}\right)^2\right] \sin^2\frac{r_A \Delta L}{2} \\
&+ 4\left[s_{23}^4 s_{13}^4 \frac{(1+r_A)^2}{(1-r_A)^4} - 2 s_{23}^4 s_{12}^2 s_{13}^2 \frac{r_\Delta r_A}{(1-r_A)^3} - c_{23}^2 s_{23}^2 c_{12}^2 s_{12}^2 \left(\frac{r_\Delta}{r_A}\right)^2\right] \sin^2\frac{(1-r_A)\Delta L}{2}.
\end{aligned}
\tag{25}
$$

The similar order ϵ^1, $\epsilon^{3/2}$, and ϵ^2 terms $P_{\mu\tau}^{(i)}$ in the oscillation probability in the $\nu_\mu \to \nu_\tau$ channel are given in Appendix C. For $\nu_\mu \to \nu_e$ channel, $P_{\mu e}^{(i)}(\delta)$ $(i = 0, 1, 3/2, 2)$ can be obtained from its T-conjugate as $P_{e\mu}^{(i)}(-\delta)$. It is then straightforward to verify that the similar perturbative unitarity relation, with the indices e being replaced by μ in (21), holds, the task we have explicitly executed.

C. Understanding some characteristic features of the perturbative formulas

We first note that the δ dependence appears only in terms with half-integral order in ϵ, and all the terms of half integral order in ϵ contains δ dependence. It can be readily understood by the theorem A in Sec. II because odd terms of s_{13} have to be half-integral order of ϵ in our $\sqrt{\epsilon}$ perturbation theory.

Despite that the Cervera *et al.* formula is the second order formula for small $s_{13} \sim \epsilon$, practically it is often used even for relatively large θ_{13}, for example, in the analysis of parameter degeneracy [24, 59]. Therefore, it is a legitimate question to ask how large the corrections terms to the formula can be for large θ_{13}. We therefore classify each term in the oscillation probabilities into the two categories, the one which exist in the Cervera *et al.* formula (denoted for brevity as the Cervera terms), and the ones which do not (non-Cervera

terms). The non-Cervera terms in each channel are the terms with factors of either s_{13}^4 or $r_\Delta s_{13}^2$. In the $\nu_e \to \nu_\mu$ ($\nu_e \to \nu_\tau$) channel they are the last two lines in (20) ((C3)), which come from the last term in (16).

One notices that the non-Cervera terms arise only from the order ϵ^2 terms. To understand this feature let us first note that there can be no non-Cervera terms of order ϵ^1 in our $\sqrt{\epsilon}$ perturbation theory. While the ϵ^1 suppression is provided by the factor of either r_Δ or s_{13}^2, they can be among the ϵ^2 terms in the ϵ perturbation theory, which is included in the Cervera terms. Therefore, the largest possible non-Cervera terms may be contained in the order $\epsilon^{3/2}$ terms. However, they *do not* exist for the following reason: The theorem A proved in Sec. II states that all the odd terms in s_{13} must be accompanied either by $\cos\delta$ or $\sin\delta$. Then, the theorem B dictates that all the δ dependent terms must have extra suppression factor $r_\Delta \equiv \Delta m_{21}^2/\Delta m_{31}^2 \sim \epsilon$. Therefore, s_{13}^3 terms in the non-Cervera part of the oscillation probability are actually of order $\epsilon^{5/2}$. As a consequence, the non-Cervera terms exist in our second-order formulas only with the suppression factors of s_{13}^4 or $r_\Delta s_{13}^2$. Hence, they are of order ϵ^2, excluding the possibility of yielding large corrections of order $\epsilon^{3/2}$ or lower from the non-Cervera terms.

V. ACCURACY OF THE APPROXIMATE FORMULA FOR LARGE θ_{13}

In this section, we examine numerical accuracy of our perturbative formulas of the oscillation probabilities computed to order ϵ^2 for θ_{13} of the order of the Chooz limit. We focus on the $\nu_e \to \nu_\mu$ channel whereas we also did the similar analysis in the $\nu_\mu \to \nu_\mu$ channel [61]. To display accuracy of the approximate formula, two typical ways of plot are available, namely, the absolute difference $|P_{e\mu}^{\text{exact}} - P_{e\mu}^{\text{2nd}}|$, and the relative difference $|P_{e\mu}^{\text{exact}} - P_{e\mu}^{\text{2nd}}|/P_{e\mu}^{\text{exact}}$, where $P_{e\mu}^{\text{exact}}$ and $P_{e\mu}^{\text{2nd}}$ denote, respectively, numerically evaluated exact oscillation probability and our approximate formula to order ϵ^2.

In Figs. 1 and 2, the absolute and the relative differences, respectively, are presented by using the color graduation plots in $E - L$ space. We note that the comparison between the Cervera *et al.* formula and the exact result was performed in [58]. We use the similar format to make easier the comparison between our and their results. A comparison between the relevant panels in Figs. 3 and 4 of [58] and Figs. 1 and 2 indicates that our large-θ_{13} correction terms improves the accuracy of the approximate formula in a wide region in $E-L$ space. It is even more so considering that our assumed value of $\sin\theta_{13}$ is about a factor of two larger than their largest value. The features of the plots with other values of δ are quite similar to those presented in Figs. 1 and 2, and hence we do not present them. For the same reason, only the case of normal mass hierarchy is shown in Fig. 2.

To see more clearly how good (or bad) are the approximations by our and the Cervera *et al.* formula, we present in Fig. 3 the oscillation probability as a function of neutrino energy calculated numerically (denoted as "exact", green dashed line), computed by using the Cervera *et al.* formula (blue dash-dotted line), computed with our perturbative formula (red solid line). The left and the right panels in Fig. 3 are for baselines $L = 1000$ km and $L = 4000$ km, for which the matter density is taken as 2.8 g/cm^3 and 3.6 g/cm^3, respectively. The mass hierarchy is taken as the normal one, $\Delta m_{31}^2 > 0$. θ_{13} is taken as $\sin\theta_{13} = 0.18$ which is close to the Chooz limit, and $\delta = 0$. The values of the remaining mixing parameters used are given in the caption of Fig. 1.

As seen in Fig. 3, the difference between the exact and the Cervera *et al.* formula is very visible (modest) at $L = 4000$ km ($L = 1000$ km). It is notable that the higher

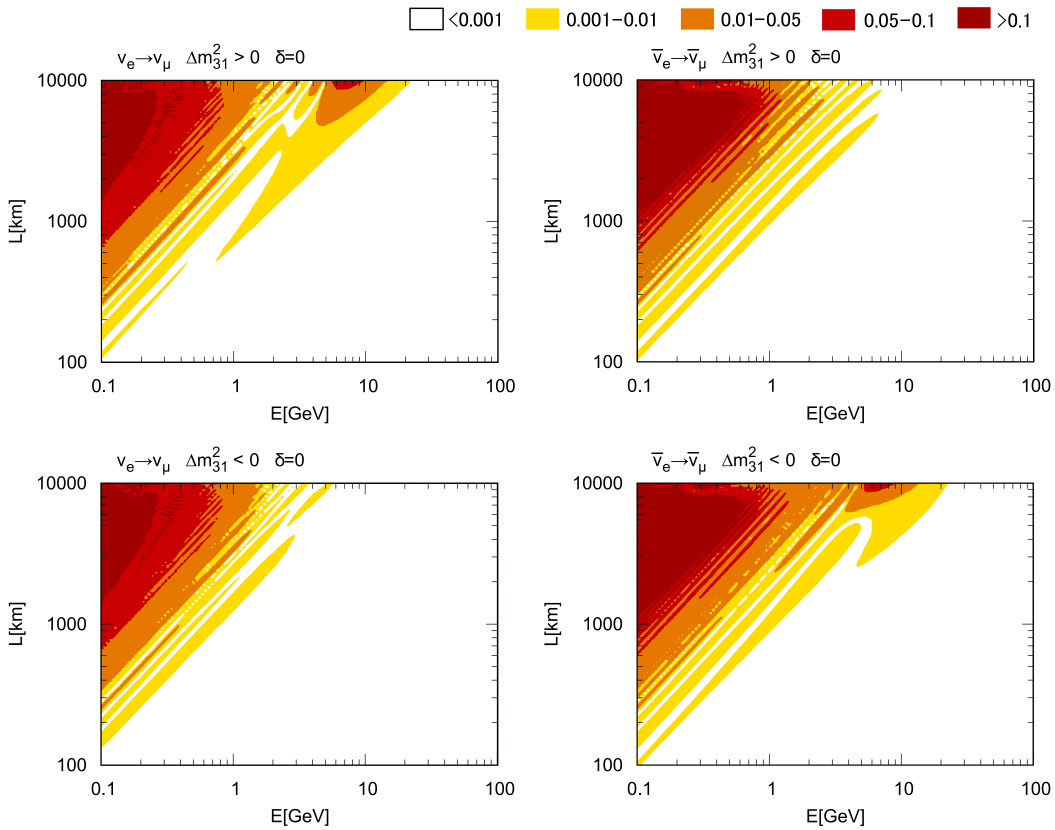

FIG. 1: The absolute difference between the numerically evaluated exact oscillation probability and our approximate formula to order ϵ^2 in the $\nu_e \to \nu_\mu$ channel, $|P^{\rm exact}_{e\mu} - P^{\rm 2nd}_{e\mu}|$, is presented by using the color graduation plot in $E-L$ space. The top (bottom) two panels are for the normal (inverted) mass hierarchy, the left neutrino and the right anti-neutrino channels. The correspondence between colors and the probability difference is given at the top of the figure. The matter density is taken as 3.0 g/cm^3 for all baselines. θ_{13} is taken as $\sin\theta_{13} = 0.18$ and $\delta = 0$. The remaining mixing parameters are chosen as $\Delta m^2_{31} = 2.4 \times 10^{-3}$ eV^2, $\Delta m^2_{21} = 7.7 \times 10^{-5}$ eV^2, $\sin^2\theta_{23} = 0.5$, and $\tan^2\theta_{12} = 0.44$.

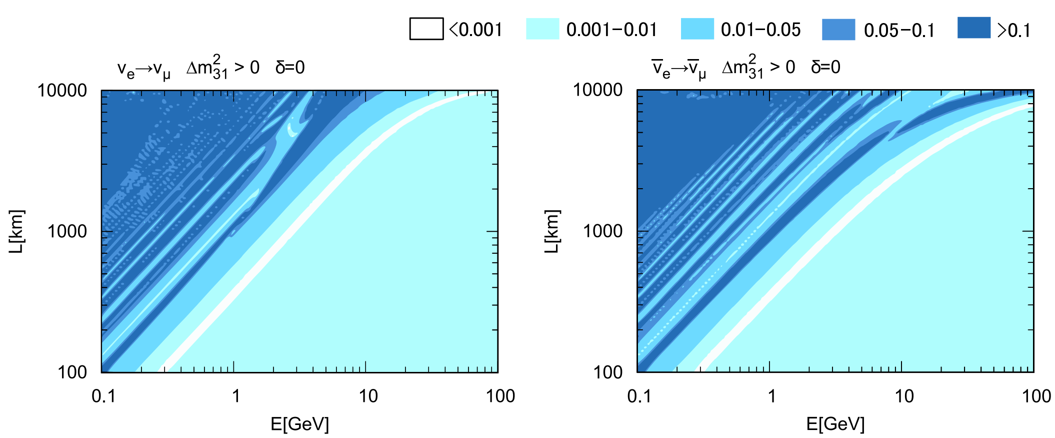

FIG. 2: The relative difference $|P^{\rm exact}_{e\mu} - P^{\rm 2nd}_{e\mu}|/P^{\rm exact}_{e\mu}$ is presented with the same format and by using the same values of the parameters as in Fig. 1.

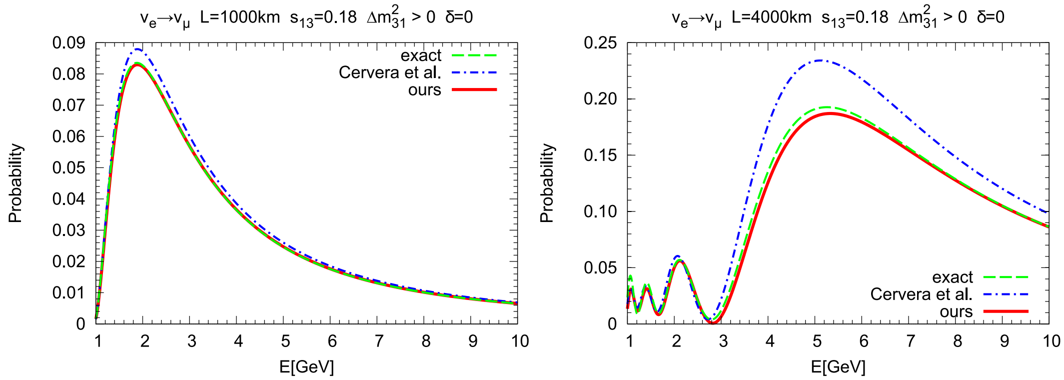

FIG. 3: Comparison between the exact oscillation probability $P(\nu_e \to \nu_\mu)$ computed numerically as a function of energy (green dashed line), the one calculated by the Cervera *et al.* formula (blue dash-dotted line), and with our formula with large θ_{13} corrections (red solid line). The left and the right panels are for baselines $L = 1000$ km and $L = 4000$ km for which the matter density is taken as 2.8 g/cm^3 and 3.6 g/cm^3, respectively. θ_{13} is taken as $\sin\theta_{13} = 0.18$ and $\delta = 0$. The values of the remaining mixing parameters are the same as given in the caption of Fig. 1.

order corrections of s_{13} incorporated in our formula nicely fill the gap between them. To understand the nature of the gap we have examined the other cases of mass hierarchy and (anti-)neutrino channel. The features of the three curves presented in Fig. 3 are very similar to the ones of antineutrino channel with the inverted hierarchy, both $r_A \equiv \frac{a}{\Delta m^2_{31}} > 0$. The features in the other two cases, antineutrino channel with the normal hierarchy and neutrino channel with the inverted hierarchy, both $r_A < 0$, are similar to each other. They are not presented because the differences among the three curves are much smaller for both $L = 1000$ km and 4000 km. We have checked that the features mentioned above are very similar for other values of δ.

As was proved in Sec. IV C the higher order corrections (the non-Cervera terms) are of order $\epsilon^2 \simeq 10^{-3}$, and hence the large gap between the exact and the Cervera *et al.* formula, in particular the one at $L = 4000$ km, must arise as a result of some enhancement. It is caused by the factor $1/(1-r_A)$, as seen from the relevant energies of the gap and the features of the other channels and hierarchies; It is nothing but the enhancement due to the MSW resonance [62, 63]. It is interesting that our formula, though perturbative, can incorporate the enhancement effect for baselines up to several thousand km for which $\Delta L \simeq 5$, not too far from $\sim \mathcal{O}(1)$. For baseline of order $L \sim 10000$ km the resonance enhancement becomes more significant and the difference between the exact and our formula (as well as the Cervera *et al.*'s) blows up. Of course, it is outside the region of validity of the perturbative treatment, and one need to sum up $1/(1 - r_A)$ effect. It is beyond the scope of this paper.

VI. $P(\nu_e \to \nu_\mu)$ TO ORDER $\epsilon^{5/2}$ AND ITS δ DEPENDENCE

For large θ_{13} comparable to the Chooz limit, it may be meaningful to analyze the order $\epsilon^{5/2}$ terms, though it has an extra suppression of $\sqrt{\epsilon} \simeq 0.2$ compared to the order $\mathcal{O}(\epsilon^2)$ terms. It could be particularly relevant for the $\nu_e \to \nu_\mu$ channel for which an extreme accuracy may be reached e.g., by neutrino factory [64]. Our interest in the $\epsilon^{5/2}$ terms is primarily

due to that they consist only of δ-dependent terms, the property enforced by the theorem A. Fortunately, it is easy to compute the order $\epsilon^{5/2}$ terms in the oscillation probability by using the $\tilde{S}$ matrix elements to order ϵ^2 listed in Appendix B.

The results of the order $\epsilon^{5/2}$ terms in $P(\nu_e \to \nu_\mu)$ which is to be added to the ϵ^1, $\epsilon^{3/2}$, and ϵ^2 terms given in Sec. IV A reads:

$$
\begin{aligned}
& P_{e\mu}^{(5/2)} = 8J_r s_{13}^2 \frac{r_\Delta r_A}{(1-r_A)^3} \cos\delta \sin^2 \frac{(1-r_A)\Delta L}{2} \\
+ \;& 8J_r \frac{r_\Delta}{r_A(1-r_A)} \left[-2s_{13}^2 \frac{r_A}{(1-r_A)^2} + (c_{12}^2 - s_{12}^2)\frac{r_\Delta}{r_A} + s_{12}^2 \frac{r_\Delta r_A}{1-r_A} \right] \\
& \qquad\qquad\qquad\qquad \times \cos\left(\delta - \frac{\Delta L}{2}\right) \sin \frac{r_A \Delta L}{2} \sin \frac{(1-r_A)\Delta L}{2} \\
+ \;& 8J_r s_{13}^2 \frac{r_\Delta}{(1-r_A)^2} (\Delta L) \cos\left(\delta - \frac{\Delta L}{2}\right) \sin \frac{r_A \Delta L}{2} \cos \frac{(1-r_A)\Delta L}{2} \\
- \;& 4J_r s_{12}^2 \frac{r_\Delta^2}{r_A(1-r_A)} (\Delta L) \cos\left(\delta - \frac{r_A \Delta L}{2}\right) \sin \frac{r_A \Delta L}{2} \\
- \;& 4J_r c_{12}^2 \frac{r_\Delta^2}{r_A(1-r_A)} (\Delta L) \cos\left(\delta - \frac{(1+r_A)\Delta L}{2}\right) \sin \frac{(1-r_A)\Delta L}{2} \\
- \;& 4J_r \frac{r_\Delta}{r_A(1-r_A)} \left(s_{13}^2 \frac{r_A}{1-r_A} - s_{12}^2 r_\Delta \right) (\Delta L) \cos\left(\delta - \frac{(1-r_A)\Delta L}{2}\right) \sin \frac{(1-r_A)\Delta L}{2}.
\end{aligned}
\tag{26}
$$

$P_{e\tau}^{(5/2)}$ can be obtained from $P_{e\mu}^{(5/2)}$ by the transformation $c_{23} \to -s_{23}$ and $s_{23} \to c_{23}$.

The distinctive feature of order $\epsilon^{5/2}$ terms in (26), as compared to $\epsilon^{3/2}$ terms in (19), is that there exist much more profound type of δ dependence, not only correlated (as argument of cosine function) with $\frac{\Delta L}{2}$ but also to $\frac{r_A \Delta L}{2}$ and $\frac{(1 \pm r_A)\Delta L}{2}$. If measurement is sufficiently accurate to resolve such correlations different from the vacuum type, it must merit to achieve higher sensitivity to detect CP violation and accurately measure δ. However, quantitative analysis to examine this feature to reveal the required experimental conditions, such as which energy resolution and how many baselines are required etc. is beyond the scope of this paper.

VII. PARAMETER DETERMINATION AND DEGENERACY

The analysis of parameter degeneracy e.g., in [24] can, in principle, be repeated with our formula with higher-order corrections of s_{13}. However, since the probability contains quartic terms of s_{13}, one has to deal with eighth-order equations of s_{13} to obtain the full degeneracy solutions, a formidable task to carry out. Fortunately, we know empirically that apparently there is no other solution beside the known eight-fold degeneracy for θ_{13} below the Chooz limit [65], though it may worth further examination. Therefore, in this paper we limit ourselves to the known degeneracy solutions and estimate corrections to them due to the large-θ_{13} correction terms.

We first discuss relationship between the determined mixing parameters with and without the non-Cervera terms. Given the "observable", a pair of the oscillation probabilities $P_{e\mu} \equiv P(\nu_e \to \nu_\mu)$ and $\bar{P}_{e\mu} \equiv P(\bar{\nu}_e \to \bar{\nu}_\mu)$ at certain neutrino energy E, we consider the problem

of determining the set of parameters ($s \equiv s_{13}, \delta$). If we use the Cervera *et al.* formula, the observable are related to mixing parameters (s_C, δ_C) as

$$
\begin{aligned}
P_{e\mu} - Z &= X_{\pm} s_C^2 + Y_{\pm} s_C \left(\cos\delta_C \cos\Delta_{31} \pm \sin\delta_C \sin\Delta_{31}\right), \\
\bar{P}_{e\mu} - Z &= X_{\mp} s_C^2 - Y_{\mp} s_C \left(\cos\delta_C \cos\Delta_{31} \mp \sin\delta_C \sin\Delta_{31}\right),
\end{aligned} \tag{27}
$$

where $\Delta_{31} \equiv |\Delta L/2|$, and X, Y, and Z are the coefficients whose explicit forms may be constructed from the formulas given in Sec. IV (or, see equation (2.7) in [24]). The $\pm$ signs in this section imply the normal and the inverted mass hierarchies.

When the non-Cervera corrections are taken into account the same observable yield slightly different set (s_T, δ_T) of mixing parameters as

$$
\begin{aligned}
P_{e\mu} - Z &= X_{\pm} s_T^2 + Y_{\pm} s_T \left(\cos\delta_T \cos\Delta_{31} \pm \sin\delta_T \sin\Delta_{31}\right) + P_{NC}, \\
\bar{P}_{e\mu} - Z &= X_{\mp} s_T^2 - Y_{\mp} s_T \left(\cos\delta_T \cos\Delta_{31} \mp \sin\delta_T \sin\Delta_{31}\right) + \bar{P}_{NC},
\end{aligned} \tag{28}
$$

where P_{NC} and $\bar{P}_{NC}$ denote the non-Cervera corrections in neutrino and anti-neutrino channels, respectively. The point is that the arguments in P_{NC} and $\bar{P}_{NC}$ is in principle (s_T, δ_T) but it can be replaced by (s_C, δ_C), the known quantities, because all the non-Cervera corrections are second order in ϵ. It is then easy to compute the difference between s_T and s_C and δ's to leading orders in ϵ as

$$
\xi_{\pm} \equiv s_T - s_C = -\frac{Y_{\mp} \sin(\delta_C \pm \Delta_{31}) P_{NC} + Y_{\pm} \sin(\delta_C \mp \Delta_{31}) \bar{P}_{NC}}{2 s_C \left[X_{\pm} Y_{\mp} \sin(\delta_C \pm \Delta_{31}) + X_{\mp} Y_{\pm} \sin(\delta_C \mp \Delta_{31})\right] \pm Y_{\pm} Y_{\mp} \sin 2\Delta_{31}}, \tag{29}
$$

$$
\eta_{\pm} \equiv \delta_T - \delta_C = \frac{\left[2 X_{\mp} s_C - Y_{\mp} \cos(\delta_C \pm \Delta_{31})\right] P_{NC} - \left[2 X_{\pm} s_C + Y_{\pm} \cos(\delta_C \mp \Delta_{31})\right] \bar{P}_{NC}}{2 s_C^2 \left[X_{\pm} Y_{\mp} \sin(\delta_C \pm \Delta_{31}) + X_{\mp} Y_{\pm} \sin(\delta_C \mp \Delta_{31})\right] \pm s_C Y_{\pm} Y_{\mp} \sin 2\Delta_{31}}. \tag{30}
$$

It should be noticed that, given $X_{\pm} \sim \mathcal{O}(1)$, $Y_{\pm} \sim \mathcal{O}(\epsilon)$, and $P_{NC} \sim \bar{P}_{NC} \sim \mathcal{O}(\epsilon^2)$, ξ and η are of order $\epsilon^{3/2}$ and $\epsilon^{1/2}$, respectively. They are small compared to s_T (or s_C) and δ which are of order $\epsilon^{1/2}$ and ϵ^0, respectively, justifying our perturbative treatment. Therefore, inclusion of the non-Cervera corrections does not produce sizable difference in the measured parameters. It is expected because of the smallness $\sim \epsilon^2$ of the non-Cervera corrections.

Now let us discuss the parameter degeneracy. Since $s_T - s_C$ and $\delta_T - \delta_C$ are small, the degeneracy solutions obtained with the Cervera *et al.* formula must give good approximations to the ones obtained with our second order formulas. Therefore, we start from them; Suppose that all the degeneracy solutions (s_{Ci}, δ_{Ci}) ($i = II, III, ...VIII$, reserving $i = I$ for the true solution) are obtained by using the Cervera *et al.* formula. They are formally given by

$$
\begin{aligned}
s_{Ci} &= f_i(s_{C1}, \delta_{C1}), \\
\delta_{Ci} &= g_i(s_{C1}, \delta_{C1}).
\end{aligned} \tag{31}
$$

The explicit expressions of the functions f and g are given in [24]. Here, we take, as a concrete example, the case of the sign-Δm^2 degeneracy whose solutions are labeled as III or IV. We mention how to extend our analysis to other types of degeneracies. Then, $s_T - s_C$, the difference between s_{13} obtained with our and the Cervera *et al.* formula, is given for the true I and the sign-Δm^2 clone solution III as

$$
\begin{aligned}
s_{TIII} - s_{CIII} &= \xi_{\mp}(s_{CIII}, \delta_{CIII}) \\
\delta_{TIII} - \delta_{CIII} &= \eta_{\mp}(s_{CIII}, \delta_{CIII})
\end{aligned} \tag{32}
$$

Since s_{CIII} and δ_{CIII} are given as functions of s_{CI} and δ_{CI}, (32) with (31) parametrically solve s_{TIII} and δ_{TIII} as functions of s_{CI} and δ_{CI}. For the intrinsic degeneracy solutions there is no need to flip the degeneracy sign in (32). For degeneracy solutions which involve flipping the octant of θ_{23}, X and Z in ξ and η must be replaced by the ones with octant flip, $X_{true} \to X_{false} = \cot^2\theta_{23} X_{true}$ and $Z_{true} \to Z_{false} = \tan^2\theta_{23} Z_{true}$, as is done in [24]. Thus, all the degeneracy solutions obtained with use of our second order formula can be obtained by using the ones with the Cervera *et al.* formula as an intermediate step. The difference between the two is small for s_{13}, order $\sim \epsilon^{3/2}$, and somewhat larger, $\sim \epsilon^{1/2}$ for δ.

VIII. INCLUDING THE NON-STANDARD INTERACTIONS

The effects of possible nonstandard interactions (NSI) that would affect neutrino propagation in matter are usually described by adding the following additional term in the tilde basis Hamiltonian

$$\tilde{H}_{NSI} = \Delta r_A \begin{bmatrix} \tilde{\varepsilon}_{ee} & \tilde{\varepsilon}_{e\mu} & \tilde{\varepsilon}_{e\tau} \\ \tilde{\varepsilon}^*_{e\mu} & \tilde{\varepsilon}_{\mu\mu} & \tilde{\varepsilon}_{\mu\tau} \\ \tilde{\varepsilon}^*_{e\tau} & \tilde{\varepsilon}^*_{\mu\tau} & \tilde{\varepsilon}_{\tau\tau} \end{bmatrix}. \tag{33}$$

The relationship between the $\tilde{\varepsilon}_{\alpha\beta}$ and the $\varepsilon_{\alpha\beta}$ $(\alpha, \beta = e, \mu, \tau)$ parameters in the flavor basis is defined in (8). We assume, following [33], the NSI elements $\varepsilon_{\alpha\beta}$ (and hence $\tilde{\varepsilon}_{\alpha\beta}$) are all of the order of ϵ. It is a legitimate assumption because NSI comes from higher dimensional operators (dimension six or higher) which receives suppression of at least $(M_W/M_{NP})^2 \sim 10^{-2}$ with new physics scale M_{NP}. It should be mentioned, however, that the current bounds on the NSI parameters are quite loose, $\lesssim 0.1 - 1$ [66].

At small $\theta_{13} \sim \epsilon$ for which the ϵ perturbation theory is applicable it was shown in [33] that inclusion of NSI elements into the second-order formula can be done just by doing replacement (with slight change in the notations from [33])

$$\begin{aligned} r_\Delta c_{12} s_{12} &\to r_\Delta c_{12} s_{12} + r_A \tilde{\varepsilon}_{e\mu} \equiv \Xi, \\ s_{13} e^{-i\delta} &\to s_{13} e^{-i\delta} + r_A \tilde{\varepsilon}_{e\tau} \equiv \Theta, \end{aligned} \tag{34}$$

and nothing else, where $\tilde{\varepsilon}_{e\mu} = (c_{23}\varepsilon_{e\mu} - s_{23}\varepsilon_{e\tau})$ and $\tilde{\varepsilon}_{e\tau} = (s_{23}\varepsilon_{e\mu} + c_{23}\varepsilon_{e\tau})$. The ν_e-related oscillation probability is independent of $\varepsilon_{\alpha\beta}$ in the $\mu - \tau$ sector as well as ε_{ee}. Then, it is an interesting question to ask how the higher order corrections of s_{13} fit in into this picture. The second-order formula for $P(\nu_e \to \nu_\mu)$ with NSI in our $\sqrt{\epsilon}$ perturbation theory can be obtained via a straightforward computation. The result is given by $P(\nu_e \to \nu_\mu)_{NSI} = P(\nu_e \to \nu_\mu)_{NSI-C} + P(\nu_e \to \nu_\mu)_{NSI-NC}$ where

$$\begin{aligned} & P(\nu_e \to \nu_\mu)_{NSI-C} \\ = \; & 4 \left| c_{23} \Xi \left(\frac{1}{r_A} \right) \sin\frac{r_A \Delta L}{2} + s_{23} e^{i\frac{\Delta}{2}} \Theta \left(\frac{1}{1 - r_A} \right) \sin\frac{(1 - r_A)\Delta L}{2} \right|^2, \end{aligned} \tag{35}$$

$$
\begin{aligned}
P(\nu_e \to \nu_\mu)_{NSI-NC} &= 4 s_{23}^2 s_{13}^2 \left[s_{13}^2 \frac{(1+r_A)^2}{(1-r_A)^4} - 2 s_{12}^2 \frac{r_\Delta r_A}{(1-r_A)^3} \right] \sin^2 \frac{(1-r_A)\Delta L}{2} \\
&+ \ 8 s_{13}^2 \frac{r_A}{(1-r_A)^2} \left[s_{23}^2 \left(\tilde{\varepsilon}_{ee} - \tilde{\varepsilon}_{\tau\tau} \right) \frac{1}{1-r_A} + c_{23} s_{23} |\tilde{\varepsilon}_{\mu\tau}| \cos \tilde{\phi}_{\mu\tau} \right] \sin^2 \frac{(1-r_A)\Delta L}{2} \\
&+ \ 2 s_{23}^2 s_{13}^2 \frac{1}{(1-r_A)^2} \left[2 s_{13}^2 \frac{r_A}{1-r_A} - s_{12}^2 r_\Delta - \left(\tilde{\varepsilon}_{ee} - \tilde{\varepsilon}_{\tau\tau} \right) r_A \right] (\Delta L) \sin(1-r_A)\Delta L \\
&- \ 8 c_{23} s_{23} s_{13}^2 |\tilde{\varepsilon}_{\mu\tau}| \frac{1}{1-r_A} \cos \left(\tilde{\phi}_{\mu\tau} + \frac{\Delta L}{2} \right) \sin \frac{r_A \Delta L}{2} \sin \frac{(1-r_A)\Delta L}{2},
\end{aligned} \tag{36}
$$

where the NSI phase $\tilde{\phi}_{\alpha\beta}$ is defined by $\tilde{\varepsilon}_{\alpha\beta} = |\tilde{\varepsilon}_{\alpha\beta}| e^{i\tilde{\phi}_{\alpha\beta}}$. Of course, $P(\nu_e \to \nu_\tau)_{NSI}$ can be obtained by the transformation $c_{23} \to -s_{23}$ and $s_{23} \to c_{23}$.

The Cervera term (35) contains the terms of order ϵ^1, $\epsilon^{3/2}$, and ϵ^2. There exist two NSI dependent terms in $P(\nu_e \to \nu_\mu)_{NSI-C}$ in (35) of order $\epsilon^{3/2}$ which are proportional to $s_{13}|\tilde{\varepsilon}_{e\tau}| \cos(\tilde{\phi}_{e\tau} + \delta)$ or $s_{13}|\tilde{\varepsilon}_{e\mu}| \cos(\tilde{\phi}_{e\mu} + \delta - \frac{\Delta L}{2})$. Therefore, there is a confusion among the three phases δ, $\tilde{\phi}_{e\mu}$, and $\tilde{\phi}_{e\tau}$ which is known to exist in the small θ regime [56, 67], but here with an amplified magnitude for large $\theta_{13} \sim \sqrt{\epsilon}$.

Notice that the NSI elements in the $\nu_\mu - \nu_\tau$ sector as well as the diagonal ones are absent in the second-order formula of ν_e-related probabilities obtained with the ϵ perturbation theory [33]. The decoupling is supported for small θ_{13} by the actual data analysis in which all the NSI elements are taken into account at the same time [56]. However, the formula (36) tells us that it is no more true at order ϵ^2 for large θ_{13} of the order of the Chooz limit. It will require reconsideration of how to determine the NSI parameters simultaneously with the ν-mass enriched standard model parameters, the problem first addressed in [33] whose elaboration is beyond the scope of this paper.

Though the decoupling between the ν_e-related NSI elements and the ones in the $\nu_\mu - \nu_\tau$ sector does not survive, a remnant still remains. The non-Cervera terms, $P(\nu_e \to \nu_\mu)_{NSI-NC}$ in (36), consist only of order ϵ^2 terms despite that the general theorems discussed in Sec. II do not appear to apply to guarantee this property for the system with NSI. It is the $\mathcal{O}(\epsilon^2)$ nature of the non-Cervera terms that leads to a remnant of decoupling, absence of ν_e-related NSI elements in the non-Cervera terms;[5] The terms induced by the substitution (34) produce only higher-order terms of order $\sim \epsilon^{5/2}$.

IX. CONCLUSION

In this paper, we have constructed a perturbative framework dubbed as "$\sqrt{\epsilon}$ perturbation theory" to systematically compute higher-order corrections of s_{13} assuming that it is large, of the order of the Chooz limit. Despite a natural expectation that it could produce sizable corrections to the Cervera *et al.* formula, we have proven that they must be small, of the order of $\epsilon^2 \simeq 10^{-3}$, where $\epsilon \equiv \Delta m_{21}^2/\Delta m_{31}^2$. Nonetheless, we have observed that the correction terms nicely fill the gap between the exact oscillation probability and the Cervera *et al.* formula at baseline of several thousand km in a limited range of energy where they have enhancement due to the resonance effect.

[5] It can be argued that this property holds without doing any computations, as was done in the ArXiv version of this paper, arXiv:1103.4387v1 [hep-ph].

Possible large value of θ_{13} may allow detection of the δ dependent terms of $\sim \mathcal{O}\left(\epsilon^{5/2}\right)$ in future super-precision measurement because they are small only by a factor of ~ 5 compared to the $\mathcal{O}\left(\epsilon^2\right)$ terms. Therefore, we have computed the terms and found that they have different δ dependence from the vacuum effect, in correlation to the matter dependent effects. However, it remains to be seen if these complicated and coexisting correlations can be resolved by actual experimental settings.

Several characteristic features of the computed results prompted us to think about some general features of the δ dependence of the oscillation probability. It resulted into the two general theorems stated and proved in Sec. II and Appendix A. One of them (Theorem A) allows to understand why half-integral order terms in ϵ are always accompanied with cosine or sine δ. While the other one (Theorem B) illuminates that the δ dependence in the oscillation probability, both cosine and sine, must be suppressed at the small mixing angle or $\Delta m^2_{21}/\Delta m^2_{31} \to 0$ limits. Both of them cooperate to illuminate some characteristic features of the perturbatively computed oscillation probabilities

We have investigated effects of the non-Cervera correction terms on parameter determination and the degeneracy. Because the correction terms are small, their effect must be small. Nonetheless, the explicitly computed corrections to the degeneracy solutions obtained with use of the Cervera *et al.* formula may be of use when they are implemented into the analysis codes such as the ones described in [68, 69].[6]

Finally, we gave a derivation of the second-order formula of the ν_e-related appearance probabilities with large θ_{13} corrections in systems with NSI effects in propagation. The result of the Cervera terms is shown to be identical with the one obtained by the replacement to the generalized variables (34), producing enhanced NSI dependent terms of $\mathcal{O}\left(\epsilon^{3/2}\right)$. The decoupling of the NSI elements in the $\nu_\mu - \nu_\tau$ sector from the ν_e-related probabilities is invalidated by the non-Cervera type corrections at order ϵ^2, which calls for reconsideration of the strategy for parameter determination.

Appendix A: Theorem B for Suppression of CP Violation for Vanishing Mixing Angle

Here, we attempt to prove the theorem B. We first describe our general strategy. We introduce a diagonal phase transformation matrix $T = \text{diag}(e^{i\alpha}, e^{i\beta}, e^{i\gamma})$, and define a "hat basis" as $\hat{\nu} = T\nu$ with the Hamiltonian $\hat{H} \equiv THT^{\dagger}$. We choose α, β, and γ such that $\hat{H}$ is free from any phases including δ. Since T is a diagonal phase transformation, it merely redefines phase of the flavor basis wave function, and hence it does not affect the probability. Therefore, if such choice of the phases is shown to be possible it implies that there is no δ dependence in the oscillation probability.

We illustrate the proof by explicitly treating the case of $\theta_{23} = 0$ case, because the situation is exactly the same in other cases. With the choice $T = \text{diag}(1, 1, e^{-i\delta})$, the hat basis

[6] We thank Patrick Huber for communications for feasibility of implementing the analytic degeneracy solutions to achieve fast search for the degenerate minima, the possibility raised in [24].

Hamiltonian can be written as $\hat{H} = \hat{H}_{vac} + \text{diag}(a/2E, 0, 0)$, where

$$\hat{H}_{vac} = \left[\begin{array}{ccc} \Delta_{21} s_{12}^2 c_{13}^2 + \Delta_{31} s_{13}^2 & \Delta_{21} c_{12} s_{12} c_{13} & -\Delta_{21} s_{12}^2 s_{13} c_{13} + \Delta_{31} c_{13} s_{13} \\ \Delta_{21} c_{12} s_{12} c_{13} & \Delta_{21} c_{12}^2 & -\Delta_{21} c_{12} s_{12} s_{13} \\ -\Delta_{21} s_{12}^2 s_{13} c_{13} + \Delta_{31} c_{13} s_{13} & -\Delta_{21} c_{12} s_{12} s_{13} & \Delta_{21} s_{12}^2 s_{13}^2 + \Delta_{31} c_{13}^2 \end{array} \right] \tag{A1}$$

showing explicitly that δ-dependence disappear from the Hamiltonian by the T transformation. Notice that here we have used the notation $\Delta_{ij} \equiv \Delta m^2_{ij}/2E$ $(ij = 21, 31)$, whose latter is different from the one in Sec. VII. (We hope that no confusion arises.) Therefore, δ goes away from the probability in the vanishing θ_{23} limit. We can repeat the similar exercise for other vanishing limits of s_{12} with T matrix $T = \text{diag}(e^{i\delta}, 1, 1)$. It is trivial to observe that there is no δ dependence if $\theta_{13} = 0$. Therefore, we have shown that δ dependence, both $\cos\delta$ and $\sin\delta$, disappears from the oscillation probability at the vanishing limit of one of the mixing angles for arbitrary matter density profiles.

To really prove the theorem B we have to show that the δ-dependence goes away in the large mixing angle limit $\theta_{ij} \to \pi/2$ $(ij = 12$ and $23)$. One can show that the similar method works for $c_{12} \to 0$ and $c_{23} \to 0$ with $T = \text{diag}(e^{i\delta}, 1, 1)$ and $T = \text{diag}(1, e^{-i\delta}, 1)$, respectively. Assuming that the oscillation probabilities in all channels can be expanded into power series of s_{ij} and c_{ij} the features stated above prove the theorem B for arbitrary matter density profiles.

It is interesting to observe that the similar method fails for the limit $c_{13} \to 0$. It is perfectly consistent with the fact the factor c_{13} is missing in the coefficient of $\cos\delta$ term in the oscillation probabilities in the $\nu_\mu - \nu_\tau$ sector [47]. However, since the suppression factor of the δ-dependent terms in the ν_e-related channels for constant matter density derived in the same reference [47] *does* contain c_{13}^2, it is likely that the theorem B can be generalized to the one with suppression factor J instead of J_r in these channels, a conjecture.

Appendix B: $\tilde{S}$ matrix elements to second order

The $\tilde{S}$ matrix elements can be computed by using (15) and are written as sums over the terms of order $\epsilon^{\frac{1}{2}}$, ϵ^1, $\epsilon^{\frac{3}{2}}$, and ϵ^2. The T-conjugate elements can be obtained by $\tilde{S}_{\beta\alpha}(\delta) = \tilde{S}_{\alpha\beta}(-\delta)$. Notice that all the elements that are not shown below, except for the T-conjugate ones, are vanishing.

$$\tilde{S}^{(1/2)}_{e\tau} = s_{13} e^{-i\delta} \frac{1}{1 - r_A} \left(e^{-i\Delta x} - e^{-ir_A \Delta x} \right) \tag{B1}$$

$$\begin{aligned} \tilde{S}^{(1)}_{ee} &= \left(s_{13}^2 \frac{r_A}{1 - r_A} - s_{12}^2 r_\Delta \right) (i\Delta x) e^{-ir_A \Delta x} + s_{13}^2 \frac{1}{(1 - r_A)^2} \left(e^{-i\Delta x} - e^{-ir_A \Delta x} \right) \\ \tilde{S}^{(1)}_{e\mu} &= -c_{12} s_{12} \left(\frac{r_\Delta}{r_A} \right) \left(1 - e^{-ir_A \Delta x} \right) \end{aligned} \tag{B2}$$

$$\begin{aligned} \tilde{S}^{(1)}_{\mu\mu} &= -c_{12}^2 r_\Delta (i\Delta x) \\ \tilde{S}^{(1)}_{\tau\tau} &= -s_{13}^2 \left(\frac{r_A}{1 - r_A} \right) (i\Delta x) e^{-i\Delta x} - s_{13}^2 \frac{1}{(1 - r_A)^2} \left(e^{-i\Delta x} - e^{-ir_A \Delta x} \right) \end{aligned} \tag{B3}$$

$$\begin{aligned}
\tilde{S}^{(3/2)}_{e\tau} &= -s^3_{13}e^{-i\delta}\frac{(1+r_A)^2}{2(1-r_A)^3}\left(e^{-i\Delta x}-e^{-ir_A\Delta x}\right) \\
&- s^3_{13}e^{-i\delta}\frac{r_A}{(1-r_A)^2}(i\Delta x)\left(e^{-i\Delta x}+e^{-ir_A\Delta x}\right) \\
&+ s^2_{12}s_{13}e^{-i\delta}\frac{r_\Delta}{1-r_A}\left[\frac{r_A}{1-r_A}\left(e^{-i\Delta x}-e^{-ir_A\Delta x}\right)+(i\Delta x)\,e^{-ir_A\Delta x}\right] \\
\tilde{S}^{(3/2)}_{\mu\tau} &= -c_{12}s_{12}s_{13}e^{-i\delta}\frac{r_\Delta}{1-r_A}\left[r_A\left(1-e^{-i\Delta x}\right)-\frac{1}{r_A}\left(1-e^{-ir_A\Delta x}\right)\right]
\end{aligned} \tag{B4}$$

$$\begin{aligned}
\tilde{S}^{(2)}_{ee} &= \left(s^2_{12}r_\Delta - s^2_{13}\frac{r_A}{1-r_A}\right)^2\frac{(i\Delta x)^2}{2}e^{-ir_A\Delta x} - s^4_{13}\frac{r_A}{(1-r_A)^3}(i\Delta x)e^{-i\Delta x} \\
&- \left[c^2_{12}s^2_{12}\left(\frac{r^2_\Delta}{r_A}\right) - s^2_{12}s^2_{13}r_\Delta\frac{1+r^2_A}{(1-r_A)^2} + s^4_{13}\frac{r_A(1+r_A)}{(1-r_A)^3}\right](i\Delta x)e^{-ir_A\Delta x} \\
&+ \left[2s^2_{12}s^2_{13}r_\Delta\frac{r_A}{(1-r_A)^3} - s^4_{13}\frac{r_A(2+r_A)}{(1-r_A)^4}\right]\left(e^{-i\Delta x}-e^{-ir_A\Delta x}\right) \\
&+ c^2_{12}s^2_{12}\left(\frac{r_\Delta}{r_A}\right)^2\left(1-e^{-ir_A\Delta x}\right)
\end{aligned} \tag{B5}$$

$$\begin{aligned}
\tilde{S}^{(2)}_{e\mu} &= c^3_{12}s_{12}\left(\frac{r^2_\Delta}{r_A}\right)(i\Delta x) - c_{12}s_{12}\left(\frac{r_\Delta}{r_A}\right)\left[s^2_{12}r_\Delta - s^2_{13}\frac{r_A}{1-r_A}\right](i\Delta x)e^{-ir_A\Delta x} \\
&- c_{12}s_{12}\left(\frac{r_\Delta}{r_A}\right)\left[\frac{1}{2}s^2_{13} + \left(c^2_{12}-s^2_{12}\right)\left(\frac{r_\Delta}{r_A}\right)\right]\left(1-e^{-ir_A\Delta x}\right) \\
&+ c_{12}s_{12}s^2_{13}\frac{r_\Delta r_A}{(1-r_A)^2}\left(e^{-i\Delta x}-e^{-ir_A\Delta x}\right)
\end{aligned} \tag{B6}$$

$$\tilde{S}^{(2)}_{\mu\mu} = c^4_{12}r^2_\Delta\frac{(i\Delta x)^2}{2} + c^2_{12}s^2_{12}\left(\frac{r^2_\Delta}{r_A}\right)(i\Delta x) - c^2_{12}s^2_{12}\left(\frac{r_\Delta}{r_A}\right)^2\left(1-e^{-ir_A\Delta x}\right) \tag{B7}$$

$$\begin{aligned}
\tilde{S}^{(2)}_{\tau\tau} &= s^4_{13}\left(\frac{r_A}{1-r_A}\right)^2\frac{(i\Delta x)^2}{2}e^{-i\Delta x} \\
&+ \left[s^4_{13}\frac{r_A(1+r_A)}{(1-r_A)^3} - s^2_{12}s^2_{13}r_\Delta\left(\frac{r_A}{1-r_A}\right)^2\right](i\Delta x)e^{-i\Delta x} \\
&+ \left[s^4_{13}\frac{r_A}{(1-r_A)^3} - s^2_{12}s^2_{13}r_\Delta\frac{1}{(1-r_A)^2}\right](i\Delta x)e^{-ir_A\Delta x} \\
&+ \left[s^4_{13}\frac{r_A(2+r_A)}{(1-r_A)^4} - 2s^2_{12}s^2_{13}r_\Delta\frac{r_A}{(1-r_A)^3}\right]\left(e^{-i\Delta x}-e^{-ir_A\Delta x}\right)
\end{aligned} \tag{B8}$$

The S matrix elements can be obtained from the $\tilde{S}$ matrix elements by

$$S \equiv \begin{bmatrix} S_{ee} & S_{e\mu} & S_{e\tau} \\ S_{\mu e} & S_{\mu\mu} & S_{\mu\tau} \\ S_{\tau e} & S_{\tau\mu} & S_{\tau\tau} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{S}_{ee} & c_{23}\tilde{S}_{e\mu} + s_{23}\tilde{S}_{e\tau} & -s_{23}\tilde{S}_{e\mu} + c_{23}\tilde{S}_{e\tau} \\ c_{23}\tilde{S}_{\mu e} + s_{23}\tilde{S}_{\tau e} & c_{23}^2\tilde{S}_{\mu\mu} + s_{23}^2\tilde{S}_{\tau\tau} + c_{23}s_{23}(\tilde{S}_{\mu\tau} + \tilde{S}_{\tau\mu}) & c_{23}^2\tilde{S}_{\mu\tau} - s_{23}^2\tilde{S}_{\tau\mu} + c_{23}s_{23}(\tilde{S}_{\tau\tau} - \tilde{S}_{\mu\mu}) \\ -s_{23}\tilde{S}_{\mu e} + c_{23}\tilde{S}_{\tau e} & c_{23}^2\tilde{S}_{\tau\mu} - s_{23}^2\tilde{S}_{\mu\tau} + c_{23}s_{23}(\tilde{S}_{\tau\tau} - \tilde{S}_{\mu\mu}) & s_{23}^2\tilde{S}_{\mu\mu} + c_{23}^2\tilde{S}_{\tau\tau} - c_{23}s_{23}(\tilde{S}_{\mu\tau} + \tilde{S}_{\tau\mu}) \end{bmatrix}. \tag{B9}$$

Appendix C: Second Order Expressions of Oscillation Probabilities

In this Appendix C we present some remaining expressions of the oscillation probabilities. In the $\nu_e \to \nu_\tau$ channel, $P_{e\tau}^{(i)}$ $(i = 1, 3/2, 2)$ are given in each order by

$$P_{e\tau}^{(1)} = 4c_{23}^2 s_{13}^2 \frac{1}{(1-r_A)^2} \sin^2 \frac{(1-r_A)\Delta L}{2}, \tag{C1}$$

$$P_{e\tau}^{(3/2)} = -8J_r \frac{r_\Delta}{r_A(1-r_A)} \cos\left(\delta - \frac{\Delta L}{2}\right) \sin\frac{r_A \Delta L}{2} \sin\frac{(1-r_A)\Delta L}{2}, \tag{C2}$$

$$\begin{aligned} P_{e\tau}^{(2)} &= 4s_{23}^2 c_{12}^2 s_{12}^2 \left(\frac{r_\Delta}{r_A}\right)^2 \sin^2\frac{r_A \Delta L}{2} \\ &- 4c_{23}^2 \left[s_{13}^4 \frac{(1+r_A)^2}{(1-r_A)^4} - 2s_{12}^2 s_{13}^2 \frac{r_\Delta r_A}{(1-r_A)^3}\right] \sin^2\frac{(1-r_A)\Delta L}{2}. \\ &+ 2c_{23}^2 \left[2s_{13}^4 \frac{r_A}{(1-r_A)^3} - s_{12}^2 s_{13}^2 \frac{r_\Delta}{(1-r_A)^2}\right] (\Delta L) \sin(1-r_A)\Delta L. \end{aligned} \tag{C3}$$

While in the $\nu_\mu \to \nu_\tau$ channel, $P_{\mu\tau}^{(i)}$ $(i = 0, 1, 3/2, 2)$ are given by

$$P_{\mu\tau}^{(0)} = 4c_{23}^2 s_{23}^2 \sin^2\left(\frac{\Delta L}{2}\right), \tag{C4}$$

$$\begin{aligned} P_{\mu\tau}^{(1)} &= 2c_{23}^2 s_{23}^2 \left[s_{13}^2 \frac{r_A}{1-r_A} - c_{12}^2 r_\Delta\right] (\Delta L) \sin\Delta L \\ &- 8c_{23}^2 s_{23}^2 s_{13}^2 \frac{1}{(1-r_A)^2} \sin\left(\frac{\Delta L}{2}\right) \cos\left(\frac{r_A \Delta L}{2}\right) \sin\frac{(1-r_A)\Delta L}{2}, \end{aligned} \tag{C5}$$

$$\begin{aligned} P_{\mu\tau}^{(3/2)} &= 8J_r \cos\delta \left(c_{23}^2 - s_{23}^2\right) \frac{r_\Delta}{1-r_A} \\ &\times \left[r_A \sin^2\left(\frac{\Delta L}{2}\right) - \frac{1}{r_A} \sin\left(\frac{\Delta L}{2}\right) \sin\left(\frac{r_A \Delta L}{2}\right) \cos\frac{(1-r_A)\Delta L}{2}\right] \\ &+ 8J_r \sin\delta \frac{r_\Delta}{r_A(1-r_A)} \sin\left(\frac{\Delta L}{2}\right) \sin\left(\frac{r_A \Delta L}{2}\right) \sin\frac{(1-r_A)\Delta L}{2}, \end{aligned} \tag{C6}$$

$$
\begin{aligned}
P_{\mu\tau}^{(2)} &= c_{23}^2 s_{23}^2 \left(s_{13}^2 \frac{r_A}{1-r_A} - c_{12}^2 r_\Delta \right)^2 (\Delta L)^2 \cos \Delta L \\
&- 2c_{23}^2 s_{23}^2 \left[s_{13}^4 \frac{r_A(1+r_A)}{(1-r_A)^3} - (c_{12}^2 + s_{12}^2 r_A^2) s_{13}^2 \frac{r_\Delta}{(1-r_A)^2} - c_{12}^2 s_{12}^2 \frac{r_\Delta^2}{r_A} \right] (\Delta L) \sin \Delta L \\
&- 2c_{23}^2 s_{23}^2 \left[s_{13}^4 \frac{r_A}{(1-r_A)^3} + (c_{12}^2 - s_{12}^2) s_{13}^2 \frac{r_\Delta}{(1-r_A)^2} \right] (\Delta L) \sin r_A \Delta L \\
&- 2c_{23}^2 s_{23}^2 \left[2s_{13}^4 \frac{r_A}{(1-r_A)^3} - s_{12}^2 s_{13}^2 \frac{r_\Delta}{(1-r_A)^2} \right] (\Delta L) \sin (1-r_A) \Delta L \\
&+ 4c_{23}^2 s_{23}^2 \left[s_{13}^4 \frac{r_A(2+r_A)}{(1-r_A)^4} - 2s_{12}^2 s_{13}^2 \frac{r_\Delta r_A}{(1-r_A)^3} - c_{12}^2 s_{12}^2 \left(\frac{r_\Delta}{r_A} \right)^2 \right] \sin^2 \frac{\Delta L}{2} \\
&- 4c_{23}^2 s_{23}^2 \left[s_{13}^4 \frac{r_A(2+r_A)}{(1-r_A)^4} - 2s_{12}^2 s_{13}^2 \frac{r_\Delta r_A}{(1-r_A)^3} + c_{12}^2 s_{12}^2 \left(\frac{r_\Delta}{r_A} \right)^2 \right] \sin^2 \frac{r_A \Delta L}{2} \\
&+ 4c_{23}^2 s_{23}^2 \left[s_{13}^4 \frac{(1+r_A)^2}{(1-r_A)^4} - 2s_{12}^2 s_{13}^2 \frac{r_\Delta r_A}{(1-r_A)^3} + c_{12}^2 s_{12}^2 \left(\frac{r_\Delta}{r_A} \right)^2 \right] \sin^2 \frac{(1-r_A)\Delta L}{2}. \quad \text{(C7)}
\end{aligned}
$$

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